

EQUIVARIANT SPECTRAL FLOW AND A LEFSCHETZ THEOREM ON ODD DIMENSIONAL SPIN MANIFOLDS

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SECTION 0: INTRODUCTION

As one of the most important theories in mathematics, Atiyah-Singer index theorems have various profound applications and consequences. At the same time, there are several ways to prove these theorems. Of particular interest is the heat kernel proof, which allows one to obtain refinements of the index theorems, i.e., the local index theorems for Dirac operators. Readers are referred to [BGV] for a comprehensive treatment of the heat kernel method on even dimensional manifolds. It is worthwhile to point out here that the heat kernel method also lead to direct analytic proofs of the equivariant index theorem for Dirac operators on even dimensional spin manifolds. Among the existing proofs we list Bismut [BV], Berline-Vergne [BV] and Lafferty-Yu-Zhang [LYZ].

The purpose of this paper is to present a heat kernel proof of an equivariant index theorem on odd dimensional spin manifolds, which is stated for Toeplitz operators.

Recall that Baum-Douglas [BD] first stated and proved an odd index theorem for Toeplitz operators using the general Atiyah-Singer index theorem for elliptic pseudo-differential operators. It is known to experts that one can give a heat kernel proof of the above mentioned odd index theorem. However, let us still give a brief description of the basic ideas. The first step is to apply a result of Booss-Wojciechowski [BW] to identify the index of the Toeplitz operator to the spectral flow of a certain family of self-dual elliptic operator with positive order. The second step is then to use the well-known relationship between spectral flows and variations of η -invariants to evaluate this spectral flow (cf. [G]).

Our proof of the equivariant odd index theorem follows the same strategy. For this purpose we need to introduce a concept of equivariant spectral flow and establish an equivariant version of the Booss-Wojciechowski theorem mentioned above. We then extend the relationship between the spectral flow and variations of η invariants to the equivariant setting. Finally, we use the local index techniques to evaluate these variations.

Among the methods of Bismut [B], Berline-Vergne [BV] and Lafferty-Yu-Zhang [LYZ], for simplicity we will follow those of [LYZ] in this paper. There is, however, no difficulty applying other methods.

Also notice that Dai and Zhang [DZ] introduced the concept of higher spectral flow and gave a heat kernel treatment to the family index problem for Toeplitz operators.

This paper is organized as follows. In Section One, we review the basic definition of the Toeplitz operators associated to Dirac operators on odd dimensional spin manifolds and prove the equivariant odd index theorem by using the Baum-Douglas [BD] trick and also the general Atiyah-Singer Lefschetz fixed point theorem [AS] for elliptic pseudo-differential operators. In Section Two, we introduce the equivariant spectral flow and prove an equivariant extension of the Booss-Wojciechowski theorem [BW]. In Section Three, we establish a relation between the equivariant spectral flow and the variations of equivariant η invariants. This in turn gives a heat kernel formula for the equivariant index of the Toeplitz operators. In Section Four, we evaluate these variations by adopting the local index theorem techniques.

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SECTION 1: TOEPLITZ OPERATORS AND A LEFSCHETZ FIX POINT THEOREM

We begin by fixing notations on odd dimensional Clifford algebras that are used in this paper. From now on, we fix $n = 2m + 1$, where m is a positive integer.

Let V be a n -dimensional real vector space associated with a positive inner product. Set e_1, \dots, e_n to be an orthonormal basis for V . Denote

$$T(V) = \mathbb{R} \oplus V \oplus V \otimes V \oplus \dots, \quad (1.1)$$

and I to be the two-sided ideal of $T(V)$ generated by $\{x \otimes x + (x, x)1; x \in V\}$. The Clifford algebra associate to V is defined as

$$C(V) = T(V)/I. \quad (1.2)$$

which is also called as $C(n)$ sometimes. It is clear that $\{c_i = e_i I \in C(V)\}$ is the set of generators of $C(V)$ satisfying the following relations:

$$c_i c_j + c_j c_i = -2\delta_{ij}. \quad (1.3)$$

Define the chirality operator of $C(n) \otimes \mathbb{C}$ to be

$$\Gamma = (\sqrt{-1})^{m+1} c_1 \dots c_n, \quad (1.4)$$

which can be checked to be in the center of $C(n) \otimes \mathbb{C}$. It is known that there is a unique irreducible complex $C(n)$ representation S of dimension 2^m , such that $\Gamma = Id_S$ on S .

For future use, we define the symbol map $\sigma : C(n) \rightarrow End(\wedge \mathbb{C}^n)$:

$$\sigma(c_i) = e_i \wedge -\iota(e_i^*), \quad (1.5)$$

which is a complex representation of $C(n)$. For $x \in C(n)$ such that x is not a scalar, we have

$$\text{Tr}_S(x) = -\sqrt{-1}(-2\sqrt{-1})^m (\sigma(x)1)_{[n]}, \quad (1.6)$$

where $(\cdot)_{[d]}$, for any integer d , denotes the d -dimensional part of an exterior form.

We proceed to define the Toeplitz operator on a closed spin manifold.

Through out this paper, we assume M to be a closed (compact, without boundary), oriented, spin manifold with dimension $n = 2m + 1$ and a fixed spin structure. We also fix a Riemannian metric g_{TM} on M .

Let $S(M)$ be the canonical complex spinor bundle of M , which is also a $C(T^*M)$ -module. Let ∇^{TM} be the canonical Levi-Civita connection, which induces a natural connection ∇^S on S . Choose a local orthonormal basis e_1, \dots, e_n is for TM , with dual basis $e^1, \dots, e^n \in T^*M$. The canonical Dirac operator on S can be defined to be

$$D^S = \sum c(e^i) \nabla_{e_i}^S. \quad (1.7)$$

It is well known that D^S is a self-adjoint first-order elliptic differential operator acting on $S(M)$. Therefore, there is a spectral decomposition of $\Gamma_{L^2}(S)$ according to D^S . Denote $L_+^2(S)$ to be the direct sum of eigenspaces of D associated to nonnegative eigenvalues, and P_+ to be the orthogonal projection operator from $L^2(S)$ to $L_+^2(S)$. Set $P_- = Id - P_+$ and

$$P = P_+ - P_-. \quad (1.8)$$

Given \mathbb{C}^N a trivial complex vector bundle over M carrying the trivial metric and connection, D and P extend trivially as operators acting on $\Gamma(S \otimes \mathbb{C}^N)$. Let $g : M \rightarrow U(N)$ be a smooth map. Then, g extends to an action on $S(M) \otimes \mathbb{C}^N$ as $Id_{S(M)} \otimes g$, which is still denoted as g for simplicity.

Definition 1.1. *Define the Toeplitz operator associated to D and g to be*

$$T_g = (P_+ \otimes Id_{\mathbb{C}^N})g(P_+ \otimes Id_{\mathbb{C}^N}) : L_+^2(S(M) \otimes \mathbb{C}^N) \rightarrow L_+^2(S(M) \otimes \mathbb{C}^N). \quad (1.9)$$

It is a classical fact that T_g is a bounded Fredholm operator between the given Hilbert spaces. Furthermore, if we define Γ_λ to be the eigenspace of D with eigenvalue λ , Γ_λ is of finite dimension for each λ .

We then describe the equivariant index problem for Toeplitz operators.

Consider H a compact group of isometries of M preserving the orientation and spin structure, hence, it also acts on $\Gamma(S(M) \otimes \mathbb{C}^N)$. Since the action of H commutes with the Dirac operator D , it also commutes with P_+ , and P . Furthermore each Γ_λ is H -invariant. But to ensure the H -invariance of Toeplitz operator, we need to make the following assumption on H :

Assumption 1.2. *For $h \in H$ and any $x \in M$,*

$$g(hx) = g(x). \quad (1.10)$$

As a consequence,

$$T_g h_{\Gamma(S(M) \otimes \mathbb{C}^N)} = h_{\Gamma(S(M) \otimes \mathbb{C}^N)} T_g. \quad (1.11)$$

Definition 1.3. Given T_g and H as above and satisfying (1.10), the equivariant index of T_g , associated with H , is defined as the following virtual representation of H in $R(H)$, the representation ring of H :

$$\text{Ind}_H(T_g) = \ker T_g - \text{coker } T_g. \quad (1.12)$$

We also denote, for any $h \in H$,

$$\text{Ind}(h, T_g) = \text{Tr}(h, \text{Ind}_H(T_g)). \quad (1.13)$$

Now an application of the general Atiyah-Singer index theorem [AS] as in Baum-Douglas [BD] gives the following

Theorem 1.4. For T_g defined as above, let F_i 's be the fixed, connected sub-manifolds of M under the action of any $h \in H$, and ν_i be the normal bundle of F_i in TM , then we have

$$\text{Ind}(h, T_g) = \sum_i \left(\frac{-\sqrt{-1}}{2\pi} \right)^{m+1-\frac{\dim F_i}{2}} (\hat{A}(F) \text{ch}(g) [\text{Pf}(2 \sin(\sqrt{-1} \frac{R^\nu(F_i) + \Theta_i}{2}))])^{-1} [F_i], \quad (1.14)$$

where under any local coordinate system, Θ_i is the logarithm of the Jacobian matrix of $h|_{\nu_i}$, R^{ν_i} is the curvature matrix of the bundle ν_i , and

$$\text{ch}(g) = \int_0^1 \text{Tr}[g^{-1} dg \exp(u(1-u)(g^{-1} dg)^2)] du \quad (1.13)$$

is the so-called odd Chern character for the differentiable map $g : M \rightarrow U(N)$.

In Section Four, we will prove a local version of Theorem 1.4.

SECTION 2: EQUIVARIANT SPECTRAL FLOW AND EQUIVARIANT INDEX PROBLEM

In this section we will introduce the equivariant spectral flow and discuss its relation with the equivariant index problem that we have set up in the previous section.

Denote $I = [0, 1]$. Let $\{D_u\}_{u \in I}$ is a continuous family of self-dual elliptic operators of the positive orders on the Hilbert space $\mathcal{H} = L^2(S \otimes \mathbb{C}^N)$. For any fixed $u \in I$, $\text{Spec} D_u$ is discrete, so we can denote the corresponding eigenspace as $\Gamma_{u,\lambda}$ for any $\lambda \in \text{Spec} D_u$. Furthermore, for any open $U \in \mathbb{R}$, define $\Gamma_{u,U} = \bigoplus_{\lambda \in U} \Gamma_{u,\lambda}$.

Recall first the usual (scalar) spectral flow according to [APS2].

Definition 2.1. For $D_u (u \in I)$ a continuous family of self-dual elliptic operator of positive order, consider the graph of $\text{Spec} D_u$:

$$\mathfrak{S} = \bigcup \text{Spec} D_u, \quad (2.1)$$

which is a closed set of $\mathbb{R} \times I$. Define the spectral flow of $\{D_u\}$ to be the intersection number of \mathfrak{S} with the line $\{-\delta\} \times I$ for a sufficiently small positive δ , which is denoted as $\text{sf}(\{D_u\})$.

Notice that if both D_0 and D_1 are invertible, we can simply replace δ in the above definition by 0.

We then would like to extend this notion to the equivariant case.

Let H be as in Section One and $R(H)$ be its representation ring. We further assume that each D_u in the above discussion is compatible with the action of H . Thus, every $\Gamma_{u,\lambda}$ can be viewed as an element of $R(H)$.

We establish the following lemma, which is an extension of continuity of the spectrums of the family of self-dual operators.

Lemma 2.2. *Let $\{D_u\}$ be described as above. For a fixed $u_0 \in I$ and any $\lambda \in \text{Spec} D_{u_0}$ with $\dim \Gamma_{u_0, \lambda} = k$, we can find a positive ϵ such that for any $u \in I$, $|u - u_0| < \epsilon$, there is an open set $U = U(u_0)$ containing λ and depending only on u_0 , such that*

$$\dim \Gamma_{u, U} = k. \quad (2.2)$$

Furthermore,

$$\Gamma_{u, U} = \Gamma_{u_0, \lambda}, \quad (2.3)$$

as elements in $R(H)$.

Proof. (2.2) is actually proved in [BW] (Lemma 17.1). More precisely, by [BW], there exist a $\epsilon > 0$ and k continuous functions

$$f_1, \dots, f_k : (u_0 - \epsilon, u_0 + \epsilon) \rightarrow \mathbb{R}, \quad (2.4)$$

such that

$$f_i(u_0) = \lambda; \quad (2.5)$$

furthermore, for any $u \in (u_0 - \epsilon, u_0 + \epsilon)$, there exists an open set U that depends only on u_0 and contains λ satisfying

$$\{f_j(u)\}_{j=1}^k = \text{Spec} D_u \cap U. \quad (2.6)$$

Let Q_u , $u_0 - \epsilon < u < u_0 + \epsilon$, be the orthonormal projections of $\Gamma_{u, U}$ onto $\Gamma_{u_0, U}$. By the continuity of $\{D_u\}$ and f_j 's, $\{Q_u\}$ is a continuous family of self-adjoint projections. Thus, it is possible to re-adjust ϵ if necessary so that

$$\|Q_u - Q_{u_0}\| < 1, \quad (2.7)$$

for $u_0 - \epsilon < u < u_0 + \epsilon$. Now using a trick of Reed and Simon [RS, p.72], if we define

$$W_u = (1 - (Q_u - Q_{u_0})^2)^{-\frac{1}{2}} [Q_u Q_{u_0} + (1 - Q_u)(1 - Q_{u_0})], \quad (2.8)$$

it is easy to verify that W_u is unitary and

$$W_u^{-1} Q_u W_u = Q_{u_0}. \quad (2.9)$$

Notice that the image of Q_u is $\Gamma_{u, U}$ and the above construction is H -compatible, (2.3) easily follows.

Now we can proceed to define the equivariant spectral flow. Given $\{D_u\}$ as above, we define

$$\text{Spec}_H D_u = \{(\lambda, \Gamma_{u, \lambda}); \lambda \in \text{Spec} D_u\}. \quad (2.10)$$

By Lemma 2.2 and the fact that $R(H)$ has only countable many irreducible elements, there exist $f_j(u) \in C(I)$ and $R_j \in R(H)$, for $j \in \mathbb{N}$, such that

$$\cup_{u \in I} \text{Spec} D_u = \cup_j (f_j(u), R_j). \quad (2.11)$$

Hence, we can introduce the following:

Definition 2.3. Given D_u as above, we define the equivariant spectral flow as

$$\text{sf}_H(\{D_u\}) = \sum_j \epsilon(f_j) R_j, \quad (2.12)$$

where $\epsilon(f_j)$ is the intersection number of the graph f_j with the line $u = -\delta$ for sufficiently small positive δ . We also denote

$$\text{sf}(h, \{D_u\}) = \text{Tr}(h, \text{sf}_H(\{D_u\})). \quad (2.13)$$

Remark 2.4. It is not hard to see that, as in the scalar case, only finite many $\epsilon(f_k)$'s in the above definition are non-zero. Also, if both D_0 and D_1 are invertible, δ can simply be replaced by 0.

Remark 2.5. As in the scalar case, the equivariant spectral flow is a homotopy invariant. In particular, let E_σ be the affine space of all the elliptic, positive-ordered operators in \mathcal{H} with the same symbol σ . Then E_σ is convex, and hence contractible. Therefore given any two fixed H -compatible points in E_σ , the equivariant spectral of two different H -compatible paths connecting them are the same.

Remark 2.6. Applying the method used above, it is not hard to extend the notion of higher spectral flow in the sense of Dai-Zhang [DZ] to the equivariant setting. We leave the details to the interested readers.

For the rest of this paper, we pick a particular family $\{D_u\}$ as below:

$$D_u = (1 - u)D + ug^{-1}Dg, \quad (2.14)$$

where D is the Dirac operator given in (1.5) and being extended to the Hilbert space $\mathcal{H} = L^2(S \otimes \mathbb{C}^N)$.

It is easy to see that this family $\{D_u\}$ satisfy the conditions in Definition 2.1 and 2.3. Furthermore, we note that D_u 's are of the same symbol, which is in turn denoted as σ .

Finally we want to prove the following theorem to clarify the relation between the equivariant spectral flow and our original index problem. The method used here is from Booss and Wojciechowski [BW].

Theorem 2.7.

$$\text{Ind}(h, T_g) = -\text{sf}(h, \{D_u\}). \quad (2.15)$$

Proof. Define

$$P_u = (1 - u)P + ug^{-1}Pg, \quad (2.16)$$

where P is defined in (1.8). Apply the same argument used in the proof of [BW] Theorem 17.17, noticing that it is compatible with our equivariant setting. Thus, we conclude that

$$\text{sf}_H(\{D_u\}) = \text{sf}_H(\{P_u\}). \quad (2.17)$$

Straightforward calculation gives that,

$$\ker(T_g) = \{u \in P_+ \mathcal{H}, gu \in P_- \mathcal{H}\},$$

$$\text{coker}(T_g) = \{u \in P_- \mathcal{H}, gu \in P_+ \mathcal{H}\}. \quad (2.18)$$

Then, it is easy to see that

$$P_u(v) = \begin{cases} v, & v \in P_+ \mathcal{H}, \quad gv \in P_+ \mathcal{H}; \\ -v, & v \in P_- \mathcal{H}, \quad gv \in P_- \mathcal{H}; \\ (1-2u)v, & v \in \ker(T_g); \\ (2u-1)v, & v \in \text{coker}(T_g). \end{cases} \quad (2.19)$$

Combining (2.17) and (2.19), (2.15) clearly follows.

SECTION 3: EQUIVARIANT SPECTRAL FLOW AND EQUIVARIANT ETA FUNCTIONS

Eta invariants first appeared in [APS1], and has a known close relation with the spectral flow(cf. [BF, G]). In this section, we extend this relation to the equivariant case.

Definition 3.1. *Let D be a self-adjoint operator on the Hilbert space \mathcal{H} . The eta function associated to D is defined to be*

$$\eta(s, D) = \sum_{\lambda \neq 0} (\text{sign} \lambda) \frac{\dim \Gamma_\lambda}{|\lambda|^s}, \quad (3.1)$$

where $\text{Re}(s)$ is large enough, λ runs over the nonzero eigenvalues of D and Γ_λ is the eigenspace of D with eigenvalue λ .

It is then clear that

$$\eta(s, D) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty \text{Tr}(De^{-tD^2}) t^{\frac{(s-1)}{2}} dt \quad (3.2)$$

holds. By a result of Bismut and Freed ([BF]), eta function of D is analytic for $\text{Re}(s) > -1/2$, in particular, we write

$$\eta(D) = \eta(0, D). \quad (3.3)$$

Furthermore, define the truncated η function, for $\epsilon > 0$, to be

$$\eta_\epsilon(s, D) = \frac{1}{\Gamma((s+1)/2)} \int_\epsilon^\infty \text{Tr}(De^{-tD^2}) t^{\frac{(s-1)}{2}} dt \quad (3.4)$$

and write

$$\eta_\epsilon(D) = \eta_\epsilon(0, D). \quad (3.5)$$

The equivariant eta function can be defined similarly:

Definition 3.2. *Let D be defined as in Definition 3.1. Furthermore, if there is compact group H acting on \mathcal{H} and D commutes with the action of H , the equivariant eta function associated to D is defined as*

$$\eta(h, s, D) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty \text{Tr}(hDe^{-tD^2}) t^{\frac{(s-1)}{2}} dt, \quad (3.6)$$

for $\text{Re}(s)$ large enough.

A regularity result of Zhang [Z] allows us to write

$$\eta(h, D) = \eta(h, 0, D). \quad (3.7)$$

We also define the truncated equivariant eta function, for an $\epsilon > 0$, to be

$$\eta_\epsilon(h, s, D) = \frac{1}{\Gamma((s+1)/2)} \int_\epsilon^\infty \text{Tr}(hDe^{-tD^2}) t^{\frac{(s-1)}{2}} dt, \quad (3.8)$$

and

$$\eta_\epsilon(h, D) = \eta_\epsilon(h, 0, D), \quad (3.9)$$

for any $h \in H$.

We then consider the variation of equivariant eta functions.

Suppose \mathcal{F} is the real Banach space of all bounded self-adjoint operators on \mathcal{H} . Let Φ be the affine space

$$\Phi = \{D^S \otimes Id_{\mathbb{C}^N} + E, E \in \mathcal{F}\}. \quad (3.10)$$

It is clear that for any u , D_u as defined in (3.8) is in Φ .

Theorem 3.3. *For an H -invariant D in Φ and any $h \in H$, define a one form $\alpha_{\epsilon, h}$ on Φ such that for $X \in T_D\Phi = \mathcal{F}$,*

$$\alpha_{\epsilon, h}(X)(D) = (\epsilon/\pi)^{\frac{1}{2}} \text{Tr}(hXe^{-\epsilon D^2}). \quad (3.11)$$

Then, $\alpha_{\epsilon, h}$ is closed and we have

$$d\eta_\epsilon(h, D) = 2\alpha_{\epsilon, h}(D). \quad (3.12)$$

proof. The proof of Theorem 3.4 is almost the same of that of Proposition 2.5 of [G], just noticing the commutativity of h and D .

We can state the main result of this section as:

Theorem 3.4. *For any H -invariant path in $\Phi = \Phi(D_0)$ connecting D_0 and D_1 as in (2.14), and any $h \in H$, we have*

$$\text{sf}(h, \{D_u\}) = - \int_\gamma \alpha_{\epsilon, h}. \quad (3.13)$$

Proof. A similar formula for the scalar case is proved in [G]. Here we imitate the method used there.

By the fact that $\alpha_{\epsilon, h}$ is closed and also Remark 2.5, both sides of (3.13) are independent of the choice of the H -invariant path $\gamma : I \rightarrow E_\sigma$ with $\gamma(0) = D_0$ and $\gamma(1) = D_1$. $\cup \text{Spec}_H(\{\gamma(u)\})$ can be written as $\cup_j (f_j(u), R_j)$ as in Section 2, where $R_j \in R(H)$ and $f_j \in C(I)$ for $j \in \mathbb{N}$.

Using a standard transversality argument, we can choose an H -invariant path γ such that for each j , the graph of f_j intersects $\{u = 0\}$ transversally. Also, by Remark 2.4, there is only finitely many nonzero $\epsilon(f_j)$'s. Without loss of generality, let them be f_1, \dots, f_k .

It is then easy to check that, for $h \in H$,

$$\text{sf}(h, \{D_u\}) = \text{sf}(h, \gamma) = \sum_{j=1}^k \epsilon(f_j) \text{Tr}(h, R_j). \quad (3.14)$$

We can calculate the truncated equivariant eta function. For any $j \in \{1, \dots, k\}$, the contribution of the $(f_j(u), R_j)$ to $\eta_\epsilon(h, \gamma(u))$ for a $h \in H$, now denoted as $S_{u,j}$, is

$$\frac{1}{\pi^{1/2}} \text{Tr}(h, R_j) \int_{\epsilon}^{\infty} f_j(u) e^{-tf_j(u)^2} t^{-1/2} dt. \quad (3.15)$$

Notice that

$$\frac{1}{\pi^{1/2}} \int_{\epsilon}^{\infty} \lambda e^{-t\lambda^2} t^{-1/2} dt \rightarrow \pm 1, \text{ if } \lambda \rightarrow 0 \pm. \quad (3.16)$$

Hence, for \tilde{u} being any zero of $f_j(u)$, let $\epsilon(\tilde{u})$ be the intersection number of $f_j(u)$ with $\{0\} \times I$ near \tilde{u} , we have

$$S_{\tilde{u}+,j} - S_{\tilde{u}-,j} = 2\epsilon(\tilde{u}) \text{Tr}(h, R_j). \quad (3.17)$$

Summing up through all the zeros of f_j , and by the fact that

$$\sum_{\{\tilde{u} \in I; f_j(\tilde{u})=0\}} \epsilon(\tilde{u}) = \epsilon(f_j), \quad (3.18)$$

we have

$$\begin{aligned} \epsilon(f_j) \text{Tr}(h, R_j) &= \frac{1}{2} \sum_{\tilde{u}; f_j(\tilde{u})=0} (S_{\tilde{u}+,j} - S_{\tilde{u}-,j}) \\ &= \frac{1}{2} \left(- \int_{\gamma} dS_{u,j} + S_{1,j} - S_{0,j} \right). \end{aligned} \quad (3.19)$$

Now summing up for all j , we are led to

$$\sum_{j=1}^k \epsilon(f_j(u)) \text{Tr}(h, R_j) = \frac{1}{2} \left(- \int_{\gamma} d\eta_\epsilon(h, *) + \eta_\epsilon(h, D_1) - \eta_\epsilon(h, D_0) \right). \quad (3.20)$$

Combine (3.12), (3.14) and (3.20), and notice that $\eta_\epsilon(h, D_1) = \eta_\epsilon(h, D_0)$, we have (3.13).

Combining Theorem 2.7 and Theorem 3.4, we have the following:

Theorem 3.5.

$$\text{Ind}(h, T_g) = \int_0^1 (\epsilon/\pi)^{\frac{1}{2}} \text{Tr}(h \dot{D}_u e^{-\epsilon D_u^2}) du. \quad (3.21)$$

Remark 3.6. Notice that the right-hand side of (3.21) is independent of the choice of ϵ , so we can use local index technique to calculate the limit of the integrand of (3.21) when ϵ tends to 0. In such a way, we obtain a local version of Theorem 1.4.

SECTION 4: A LEFSCHETZ THEOREM ON ODD SPIN MANIFOLDS

In this section we apply the setting of [LYZ] to compute the right-hand side of (3.21).

First of all, we prove a Lichnerowitz type formula for D_u^2 .

Lemma 4.1. *We have*

$$D_u^2 = -\Delta + \frac{K}{4} + u^2 c(\omega^2) + u(-\iota(\omega^*)\nabla^S + c(d\omega) + d^*\omega), \quad (4.1)$$

where

$$\omega = \dot{D}_u = g^{-1}dg, \quad (4.2)$$

and K is the scalar curvature of M .

Proof. (4.1) follows easily from Prop 3.45 of [BGV] and the standard Lichnerowitz formula.

For a fixed $h \in H$, let $F = \{x \in M; hx = x\}$ be the fixed point set of h . Without loss of generality we assume F is a connected odd-dimensional totally geodesic submanifold and define its dimension to be k . ν be the normal bundle of F in TM , with dimension $2s$. Define $\nu(\delta) = \{x \in \nu; ||x|| < \delta\}$. Thus, ν to be invariant under h_{TM} , and $h_{TM}|_\nu$ is non-degenerate.

If $P_\epsilon(x, y) : (S \otimes \mathbb{C}^n) \rightarrow (S \otimes \mathbb{C}^n)$ is the kernel for the operator

$$O_\epsilon = (\epsilon/\pi)^{\frac{1}{2}} \int_0^1 (\dot{D}_u e^{-\epsilon D_u^2}), \quad (4.3)$$

by the standard heat equation argument, we have

$$\text{Tr}(hO_\epsilon) = \int_M \text{Tr}(hP_\epsilon(hx, x)) d\text{vol}. \quad (4.4)$$

A routine argument using pseudo-differential operators shows that if $hx \neq x$, we have

$$\lim_{\epsilon \rightarrow 0} \text{Tr}(hP_\epsilon(hx, x)) = 0. \quad (4.5)$$

As a result, we may localize the computation to F .

Notice that both \dot{D}_u and h are bounded operators, working in a local trivialization of $\nu(\delta)$ yields the following:

Lemma 4.2. *Define, for $x \in F$,*

$$L_{loc}(x) = \lim_{\epsilon \rightarrow 0} \left\{ \int_{\nu_\delta|x} \text{Tr}(hP_\epsilon(y, hy)) dy \right\} d\text{vol}_F, \quad (4.6)$$

then it exists and is independent of δ . Furthermore,

$$\text{Ind}(h, T_g) = \int_F L_{loc}(x). \quad (4.7)$$

In the remaining part of this section, we calculate $L_{loc}(x)$ for $x \in F$.

Fix any $x_0 \in F$, let $e_1, \dots, e_n \in TM$ be a local coordinate system in a neighborhood \mathcal{N} of x_0 such that e_i 's are orthonormal at x_0 and are parallel along geodesics through x_0 . And $e_1, \dots, e_k \in TF$ and $e_{k+1}, \dots, e_n \in \nu(F)$. For any $x \in \mathcal{N}$ such that $hx \in \mathcal{N}$, there is a $n \times n$ -matrix $\mathcal{J}(x)$ satisfying

$$h|_{TM}(e_1(x), \dots, e_n(x)) = (e_1(hx), \dots, e_n(hx))\mathcal{J}(x), \quad (4.8)$$

while

$$\mathcal{J}(x) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \exp(\Theta(x)) \end{pmatrix}, \quad (4.9)$$

with $\Theta(x) \in so(2s)$.

Also, denote R^{TM} to be the curvature matrix of the Levi-Civita connection on TM with respect to the chosen $\{e_i\}$:

$$(R^{TM})_{ij} = -\frac{1}{2} \sum_{p,q=1}^n R_{ijpq} e^k e^l, \quad (4.10)$$

for $1 \leq i, j \leq n$, where e^k is the dual vector of e_k . Also, if we choose the metrics and connections on TF and $\nu(F)$ to be the restrictions of those of TM , respectively, we have the following curvature matrix

$$(R^{TF})_{ij} = -\frac{1}{2} \sum_{p,q=1}^k R_{ijpq} e^p e^q, \quad (4.11)$$

for $1 \leq i, j \leq k$ and

$$(R^{\nu(F)})_{ij} = -\frac{1}{2} \sum_{p,q=1}^k R_{ijpq} e^p e^q, \quad (4.12)$$

for $k+1 \leq i, j \leq n$.

It is known that $\mathcal{J}(x)$ is invariant along the fiber of ν [BGV]. Hence, we can fix e_i 's such that we can have the following:

$$\Theta(x_0) = \begin{pmatrix} 0 & \theta_1 & & \\ -\theta_1 & 0 & & \\ & & \ddots & \\ & & & 0 & \theta_s \\ & & & -\theta_s & 0 \end{pmatrix}, \quad (4.13)$$

with $0 < \theta_i < 2\pi$,

$$R^\nu(x_0) = \begin{pmatrix} 0 & v_1 & & \\ -v_1 & 0 & & \\ & & \ddots & \\ & & & 0 & v_s \\ & & & -v_s & 0 \end{pmatrix}, \quad (4.14)$$

and

$$R^F(x_0) = \begin{pmatrix} 0 & & & & & \\ & 0 & u_1 & & & \\ & -u_1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & u_{(k-1)/2} \\ & & & & -u_{(k-1)/2} & 0 \end{pmatrix}. \quad (4.15)$$

Here all u_i 's and v_i 's are two forms representing Chern roots.

It is easy to see that the kernel of $\sigma(hO_\epsilon)$ is $\sigma(hP_\epsilon)$, where the symbol map σ is defined in (1.5).

Now we re-scale T^*M as in [BGV] to get, for $\epsilon \rightarrow 0$,

$$\begin{aligned} L_0 &= \lim_{\epsilon \rightarrow 0} \sigma(hO_\epsilon) \\ &= \int_0^1 h\left(\frac{1}{\pi}\right)^{1/2} \omega \exp\left(\sum_i (\partial_i - 1/4 \sum_j R_{ij}^{TM} b_j)^2 + u(1-u)\omega^2\right) du, \end{aligned} \quad (4.16)$$

where b_i 's are local coordinate functions on TM with respect to the chosen local charts.

We proceed as in [LYZ] to get

$$\begin{aligned} Q_0(x_0, b) &= \lim_{\epsilon \rightarrow 0} \sigma(hP_\epsilon(x, hx)) \\ &= \int_0^1 \left(\frac{1}{\pi}\right)^{1/2} [\omega \exp(u(1-u)\omega^2)] \left(\frac{1}{4\pi}\right)^{n/2} (-1)^s \\ &\quad \times \left(\prod_1^s \sin \frac{\theta_i}{2}\right) j_V(R^F) \exp\left(\frac{-1}{4} \sum_1^s \sin(\theta_i) v_i (b_{k+2i-1}^2 + b_{k+2i}^2)\right) j_V(R^\nu) \\ &\quad \times \exp\left(\sum_1^s \left(-\sqrt{-1} \frac{v_i}{2} \sin^2 \frac{\theta_i}{2} \coth\left[\sqrt{-1} \frac{v_i}{2} (b_{k+2i-1}^2 + b_{k+2i}^2)\right]\right)\right) du \\ &= \int_0^1 (-1)^s \frac{1}{2^n} \frac{1}{\pi^{m+1}} [\omega \exp(u(1-u)\omega^2)] j_V(R^F)(x) \\ &\quad \times \prod_1^s \left\{(-1) \sin\left(\frac{\theta_i}{2}\right) \frac{v_i/2}{\sin v_i/2} \exp\left[-\sin\left(\frac{\theta_i}{2}\right) \frac{v_i/2}{\sin v_i/2} \sin\left(\frac{v_i + \theta_i}{2}\right) (b_{k+2i-1}^2 + b_{k+2i}^2)\right]\right\} du \end{aligned} \quad (4.17)$$

where

$$j_V(R^\nu) = \prod \frac{\sqrt{-1} v_i/2}{\sinh(\sqrt{-1} v_i/2)}, \quad (4.18)$$

and

$$j_V(R^F) = \prod \frac{\sqrt{-1} u_i/2}{\sinh(\sqrt{-1} u_i/2)}. \quad (4.19)$$

In order to calculate $L_{loc}(x_0)$, we first apply (1.6) to $Q_0(x_0, b)$ and take the trace over \mathbb{C}^N ; then we integrate over $\nu(F)_{x_0}$. Asymptotically, that is to integrate over all b_i 's for $k+1 \leq i \leq n$. Thus, we get L_{loc} represented as a k -form on F as following

$$L_{loc}(x_0) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{m+1} (\text{ch}(g)j_V(R^F) \prod_1^s [-\pi \sin(\frac{v_i + \theta_i}{2})^{-1}])_{[k]}, \quad (4.20)$$

where $\text{ch}(g)$ is defined as in (1.13).

Here notice that since every θ_i is non-zero, $\sin(\theta_i + v_i)^{-1}$ makes sense as a polynomial expansion.

It is then easy to get the characteristic class representation from (4.20), which is the following

Theorem 4.3. *Let notations be as above. We have*

$$L_{loc}(x_0) = \left(\frac{-\sqrt{-1}}{2\pi}\right)^{m+1-s} (\hat{A}(F) \text{ch}(g) [\text{Pf}(2 \sin(\sqrt{-1}(R^\nu + \Theta)/2)]^{-1})_{[k]}(x_0). \quad (4.21)$$

Remark 4.4. *It is not hard to see that the method we have applied can also be used to prove similar local Lefschetz fixed point formulae for the Toeplitz operators associated to any Dirac-type operators.*

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